

Spectroscopy of the rotating BTZ black hole via adiabatic invariance

Xian-Ming Liu^{1,2)}

*1)Department of Physics, Hubei University for Nationalities,
Enshi Hubei, 445000, China; 2)Department of Physics,
Institute of Theoretical Physics, Beijing Normal University, Beijing, 100875, China*

Xiao-Xiong Zeng^{1,2)}

*1)Department of Physics and Engineering Technology, Sichuan University of Arts and Science,
Dazhou Sichuan, 635000, China; 2)Department of Physics,
Institute of Theoretical Physics, Beijing Normal University, Beijing, 100875, China*

Wen-Biao Liu (corresponding author)*

Department of Physics, Institute of Theoretical Physics, Beijing Normal University, Beijing, 100875, China

According to Bohr-Sommerfeld quantization rule, an equally spaced horizon area spectrum of a static, spherically symmetric black hole was obtained under an adiabatic invariant action. This method can be extended to the rotating black holes. As an example, we apply this method to the rotating BTZ black hole and obtain the quantized spectrum of the horizon area. It is shown that the area spectrum of the rotating BTZ black hole is equally spaced and irrelevant to the rotating parameter, which is consistent with the Bekenstein conjecture. Specifically, the derivation do not need the quasinormal frequencies and the small angular momentum limit.

Keywords: the rotating BTZ black hole, area and entropy spectrum, adiabatic invariance

PACS numbers: 04.70.Dy, 04.70.-s

I. INTRODUCTION

The quantization of the black hole horizon area has been the focus of theoretical physicists since the Bekenstein conjecture was proposed [1]. Supposing that the horizon area of a black hole behaves as a classical adiabatic invariant, Bekenstein showed that the quantized area spectrum has the following form

$$(\Delta A)_{\min} = 8\pi l_p^2, \quad (1)$$

where $l_p = (\frac{G\hbar}{c^3})^{1/2}$ is the Planck length. Subsequently many attempts have been made to derive the area spectrum and entropy spectrum directly utilizing the dynamical modes of this classical theory [2–6].

Hod [7] found that if one employs the correspondence principle of Bohr, the quantized area spectrum can be determined by the real part of quasinormal frequencies of the black hole. Hod suggested that the area spectrum is

$$(\Delta A)_{\min} = 4 \ln 3 l_p^2. \quad (2)$$

Based on the Bekenstein proposal for the adiabaticity of the black hole horizon area and the proposal suggested by Hod regarding the quasinormal frequencies, Kunstatter [8] derived the area spectrum of d -dimensional spherically symmetrical black holes. The specific result for the horizon area quantum is same as obtained by Hod [7] and by Bekenstein and Mukhanov [2].

The work done by Hod is an important step in this direction. However, Maggiore [9] suggested that there are some difficulties in it. Firstly the horizon area quantum originated from the real part of quasinormal frequencies is not universal. Secondly, Hod only considered transitions from the ground state to a state with large n . Once considering the general transitions $n \rightarrow n'$, the area changes only an arbitrarily incremental amount. Taking the difficulties into account, Maggiore proposed a new interpretation of the black hole quasinormal frequencies in connection to the quantum of black hole horizon area. Maggiore stated that a perturbed Schwarzschild black hole has to be regarded as a damped harmonic oscillator in which the frequency should contain both real and imaginary parts. Additionally, Maggiore showed that the most interesting case is the highly excited quasinormal modes in which the imaginary part is dominant compared to the real part. The result of Maggiore in the area spectrum is also identical to the Eq. (1) as shown by Bekenstein. Currently, the proposal of Maggior has been widely used to study the area spectra for various black holes [10–15].

*Electronic address: wbliu@bnu.edu.cn

Recently, Majhi and Vagenas [16] quantized the horizon area of a static, spherically symmetric black hole using an adiabatic invariant $\oint pdq$ and the rule of Bohr-Sommerfeld quantization $\oint pdq = 2\pi n$, where p is the conjugate momentum of the coordinate q . The resulting black hole entropy spectrum is equally spaced and the corresponding horizon area quantum is identical as derived by Maggiore [9] and by Bekenstein [1]. It is interesting that the quasinormal frequencies are not necessary to obtain the fancy spectrum in this approach. Actually, the improved Bohr-Sommerfeld quantization rule $\oint pdq = 2\pi(n + 1)$ inspired by Laudau and Lifshitz has been discussed very early by Liu et al. [17]. It was applied to a Schwarzschild black hole and the area spectrum they got is $(\Delta A) = \frac{8\pi l_p^2}{3}$. The improved quantization rule was used to explain dark matter as a remnant of black hole evaporation, where the ground state may contain an on-vanishing energy [17]. However, the area spectrum is not consistent with the following references [2–8] and the Bekenstein conjecture [1].

Let us quantize the horizon area of a rotating BTZ black hole. It should be noted that when an adiabatic invariant is an action variable of a classical system, the Bohr-Sommerfeld quantization is applied as follows:

$$I_v = \oint pdq = 2\pi(n + \frac{1}{2}).$$

Thus, we can quantize the horizon area of a rotating BTZ black hole with an action variable rather than an adiabatic invariant. It is well known that in analytical mechanics, the action I , action variable I_v , and the Hamiltonian H of any single periodic system should satisfy the relation

$$I = I_v - \int H d\tau.$$

As the action and Hamiltonian are given, the action variable can be quantized. We find that the action variable is only the black hole entropy, and the entropy and horizon area of a rotating BTZ black hole can be quantized.

The remainder of this paper is arranged as follows. In Sec. II, the rotating BTZ black hole is introduced. In the Euclidean-Kruskal form of the rotating BTZ black hole, there is a cyclic or single periodic system with period $\frac{2\pi}{\kappa_+}$, where κ_+ is the surface gravity of the event horizon. In Sec. III, the area spectroscopy in the rotating BTZ black hole is obtained.

II. REVIEW OF A ROTATING BTZ BLACK HOLE

The rotating BTZ black hole is a solution of the $(2 + 1)$ -dimensional Einstein gravity with a negative cosmological constant $1/l^2$. The corresponding line element is [18]

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2(d\phi + N^\phi(r)dt)^2, \quad (3)$$

where the squared lapse $N(r)$ and the angular shift $N^\phi(r)$ are given as

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\phi = -\frac{J}{2r^2},$$

with $-\infty < t < +\infty$, $0 < r < +\infty$, and $0 \leq \phi \leq 2\pi$. Here M and J are the standard ADM mass and angular momentum of the black hole respectively.

The rotating BTZ black hole has two horizons

$$r_\pm^2 = \frac{1}{2}Ml^2(1 \pm \Delta), \quad \Delta = [1 - (\frac{J}{Ml})^2]^{\frac{1}{2}}, \quad (4)$$

where r_+ , r_- are the outer (event) and inner (cauchy) horizon, respectively. In particular, the existence of an event horizon means a bound on the angular momentum J as $|J| \leq Ml$.

For convenience, we list Hawking temperature T_H , horizon area A_H , and angular velocity Ω_H at the event horizon as

$$T_H = \frac{r_+^2 - r_-^2}{2\pi r_+ l^2} = \frac{M\Delta}{2\pi r_+}, \quad A_H = 2\pi r_+, \quad \Omega_H = \frac{J}{2r_+^2}. \quad (5)$$

These physical quantities can also be found in many references investigating Hawking effect of the BTZ black hole [19–23].

To avoid the dragging effect in a rotating black hole, one should perform the reputed dragging coordinate transformation as

$$d\phi = -N^\phi dt = \frac{J}{2r^2} dt = \Omega dt, \quad (6)$$

where Ω is the dragged angular velocity of the rotating BTZ black hole. Substituting Eq.(6) into Eq.(3), the line element can be changed into the 2-dimensional form as

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2. \quad (7)$$

Thus the two-dimensional Euclidean metric reads

$$ds_E^2 = N^2(r)d\tau^2 + N^{-2}(r)dr^2, \quad (8)$$

where we have used the Euclidean time $\tau = -it$.

After the definition of tortoise coordinate

$$\frac{dr^*}{dr} = N^{-2}(r),$$

or

$$r^* = r + \frac{1}{2\kappa_+} \ln \frac{r - r_+}{r_+} - \frac{1}{2\kappa_-} \ln \frac{r - r_-}{r_-},$$

where $\kappa_{\pm} = \frac{M_{\pm}}{r_{\pm}}$ is the surface gravity on the outer (inner) horizon, we can obtain the Euclidean Kruskal section of the rotating BTZ black hole. Similar to the Schwarzschild black hole, this section is a cyclic or single periodic system, whose metric reads

$$ds_E^2 = N^2(r)e^{-2\kappa_+ r^*} (dT^2 + dR^2), \quad (r > r_+) \quad (9)$$

where

$$T = \frac{1}{\kappa_+} e^{\kappa_+ r^*} \sin \kappa_+ \tau, \quad (10)$$

$$R = \frac{1}{\kappa_+} e^{\kappa_+ r^*} \cos \kappa_+ \tau. \quad (11)$$

It can be readily seen that for both T and R in Eqs. (10) and (11) are the periodic function of τ with period $\frac{2\pi}{\kappa_+}$. This period is critical for Hawking temperature research via the temperature Green function[24]. In next section, we will find that it is also essential to derive the area and entropy spectrum of the rotating BTZ black hole.

III. AREA SPECTRUM OF THE ROTATING BTZ BLACK HOLE

As is known in classical mechanics [25], one can define a quantity called as the action variable for a single periodic system

$$I_v = \oint p dq,$$

where p is the conjugate momentum of the coordinate q . It can be noted that the action I , the action variable I_v , and the Hamiltonian H of any single periodic system should satisfy the relation

$$I = I_v - \int H d\tau. \quad (12)$$

Here we prefer to use the Euclidean-like space-time Eq. (8), so we use the Euclidean time coordinate τ . Based on Eq. (12), it has been found that there is a ground state energy for the Schwarzschild black hole, which may be regarded as a candidate of dark matter [17].

Again, it can be viewed that in the dragged coordinate system, the coordinate ϕ does not appear in the line element expressions as Eqs. (7), (8), and (9). It means that ϕ is an ignorable coordinate in the Lagrangian function L . To eliminate this freedom completely, the action I can be written as

$$I = \int (L - p_\phi \dot{\phi}) d\tau = \int \int_{(0,0)}^{(p_r, p_\phi)} (\dot{r} dp'_r - \dot{\phi} dp'_\phi) \frac{dr}{\dot{r}}, \quad (13)$$

where the dot indicates differentiation with respect to the Euclidean time τ , p_r and p_ϕ are the canonical momentums conjugate to r and ϕ , respectively. This action I has been used to investigate Hawking radiation of the rotating BTZ black hole in Rfs.[22, 23]. Therefore, substituting Eq.(13) into Eq.(12), the invariant action variable I_v can be expressed as

$$I_v = \int H d\tau + \int \int_{(0,0)}^{(p_r, p_\phi)} (\dot{r} dp_r' - \dot{\phi} dp_\phi') \frac{dr}{\dot{r}}. \quad (14)$$

To remove the momentum in favor of energy, we can make use of the Hamiltonian

$$\dot{r} = \frac{dH}{dp_r'} \big|_{(r, \phi, p_\phi)} = \frac{dM'}{dp_r'},$$

$$\dot{\phi} = \frac{dH}{dp_\phi'} \big|_{(\phi, r, p_r)} = \Omega \frac{dJ'}{dp_\phi'},$$

so the invariant action variable I_v can be rewritten as

$$I_v = \int \int_0^H dH' d\tau + \int \int_{(0,0)}^{(M, J)} [dM' - \Omega dJ'] \frac{dr}{\dot{r}}. \quad (15)$$

To obtain \dot{r} , one should consider radial, null geodesics for an outgoing particle. This method has been used to study Hawking tunneling radiation widely. Considering Eq.(8), the radial null paths ($ds_E^2 = 0$) can now be written as

$$\dot{r} \equiv \frac{dr}{d\tau} = \pm i N^2(r) \equiv R_\pm(r), \quad (16)$$

where ' \pm ' denotes the outgoing (incoming) radial null paths. Henceforth, our subsequent analysis will focus on the outgoing paths, since these equations are related to the quantum mechanically nontrivial features under consideration.

Thinking of Eq.(16), we have

$$\begin{aligned} \int \int_0^H dH' d\tau &= \int \int_0^H dH' \frac{dr}{R_+(r)} = \int \int_0^H dH' \frac{dr}{\dot{r}} \\ &= \int \int_{(0,0)}^{(M, J)} [dM' - \Omega dJ'] \frac{dr}{\dot{r}}, \end{aligned}$$

where in the last step we have used the energy conservation condition $dH' = dM' - \Omega dJ'$, and thus the adiabatic invariant quantity given by Eq.(15) should read

$$I_v = 2 \int \int_0^H dH' d\tau. \quad (17)$$

Later, we perform the τ -integration. According to the discussion in Sec. II, we note that τ has periodicity $\frac{2\pi}{\kappa_+}$. However, since we are considering only the outgoing paths, the integration limit for τ will be $[0, \frac{\pi}{\kappa_+}]$ and hence the adiabatic invariant quantity reads

$$I_v = 2\pi \int_0^H \frac{dH'}{\kappa_+}. \quad (18)$$

Considering $T_H = \frac{\kappa_+}{2\pi}$, the adiabatic invariant quantity given in Eq. (18) becomes

$$I_v = \int_0^H \frac{dH'}{T_H} = S_H, \quad (19)$$

where the first law of the rotating BTZ black hole thermodynamics has been used.

Finally, implementing the Bohr-Sommerfeld quantization rule

$$I_v = \oint pdq = 2\pi n,$$

we get the black hole entropy spectrum

$$S_H = 2\pi(n + \frac{1}{2}), \quad n = 1, 2, 3, \dots \quad (20)$$

Then the spacing in the entropy spectrum can be given by

$$\Delta S_H = 2\pi(n + 1 + \frac{1}{2}) - 2\pi(n + \frac{1}{2}) = 2\pi. \quad (21)$$

It appears that the entropy of the rotating BTZ black hole are discrete and the spacing is equidistant. Recalling that the black hole entropy is proportional to the black hole horizon area (since in BTZ units $8\hbar G = 1$)

$$S_H = \frac{A}{4l_p^2}, \quad (22)$$

and if we employ the spacing of the entropy spectrum given in Eq.(21), the quantum of the black hole horizon area can be

$$\Delta A = 8\pi l_p^2 = \pi\hbar, \quad (23)$$

which is similar to the Bekenstein Eq.(1).

IV. CONCLUSIONS

With the help of Bohr-Sommerfeld quantization rule, we have derived the quantized spectrum of the horizon area for a rotating BTZ black hole utilizing a new approach in the context of adiabatic invariant quantities. Our proposal is mainly based on that the Euclidean Kruskal section of the rotating BTZ black hole is a cyclic or single periodic system. The first law of black hole thermodynamics is also used in the derivation. It is shown that the entropy spectrum of the rotating BTZ black hole is equally spaced while the quantum of the horizon area identical to the results of Bekenstein. Apparently, our result is also consistent with the initial proposal of Bekenstein in that the area spectrum is independent on the black hole parameters. Specifically, our analysis does not need the black hole quasinormal frequencies, which had been used to obtain the horizon area

spectra of the rotating BTZ black hole [13]. Moreover, the small angular momentum limit for the rotating black hole, which has been used by Daghighi et al. [15] to obtain the horizon area spectra of the Kerr black hole is not necessary here.

Acknowledgments

This research is supported by the National Natural Science Foundation of China (Grant Nos.10773002, 10875012, 11175019). It is also supported by the Fundamental Research Funds for the Central Universities under Grant No.105116. Liu Xianming is also partly supported by the Team Research Program of Hubei University for Nationalities (NO. MY2011T006).

-
- [1] Bekenstein J D. The quantum mass spectrum of the Kerr black hole. *Lett Nuovo Cimento*, 1974, 11, 467-470;
Bekenstein J D. Quantum black holes as atoms, arXiv: gr-qc/9710076;
Bekenstein J D. Black holes: classical properties, thermodynamics and heuristic quantization, arXiv: gr-qc/9808028.
 - [2] Bekenstein J D, Mukahnov V F. Spectroscopy of the quantum black hole. *Phys Lett B*, 1995, 360: 7-12.
 - [3] Louko J, Makela J. Area spectrum of the Schwarzschild black hole. *Phys Rev D*, 1996, 54: 4982-4996;
J. Makela, Black hole spectrum: continuous or discrete ? . *Phys Lett B*, 1997, 390: 115-118.
 - [4] Dolgov A D, Khriplovich I B. Properties of the quantized gravitating dust shell. *Phys Lett B*, 1997, 400:12-14;
Peleg Y. The spectrum of quantum dust black holes. *Phys Lett B*, 1995, 356: 462-465.
 - [5] Barvinsky A, Das S, Kunstatter G. Spectrum of charged black holes-the big fix mechanism revisited. *Class Quantum Grav*, 2001,18:, 4845-4861;
Barvinsky A, Das S, Kunstatter G. Quantum mechanics of charged black holes. *Phys Lett B*, 2001, 517: 415-420.
 - [6] Kastrup H A. The quantum levels of isolated spherically symmetric gravitational systems. *Phys Lett B*, 1996 ,385: 75-80.
 - [7] Hod S. Bohr's correspondence principle and the area spectrum of quantum black holes. *Phys Rev Lett*, 1998, 81: 4293-4296;
Hod S. Best approximation to a reversible process in black-hole physics and the area spectrum of spherical black holes. *Phys Rev D*, 1998, 59: 024014.
 - [8] Kunstatter G. D-dimensional black hole entropy spectrum from quasinormal modes. *Phys Rev Lett*, 2003,90: 161301.
 - [9] Maggiore M. Physical interpretation of the spectrum of black hole quasinormal modes. *Phys Rev Lett*, 2008, 100: 141301.

- [10] Wei S W, Li R, Liu Y X, et al. Quantization of black hole entropy from quasinormal modes. *J High Energy Phys*, 2009,03: 076 (2009);
Wei S W, Liu Y X, Yang K, et al. Entropy/area spectra of the charged black hole from quasinormal modes. *Phys Rev D*, 2010, 81: 104042;
Ren J R, Song S X, Wei S W. Area spectrum of d-dimensional large Schwarzschild-AdS black hole from asymptotic quasinormal modes. *Commun Theor Phys*, 2010, 53:1097-1099;
Wei S W, Liu Y X, Zhao Z H, et al. Area spectrum of Schwarzschild black hole inspired by noncommutative geometry, arXiv:hep-th/1004.2005.
- [11] Fernando S. Spinning dilaton black holes in 2+1 dimensions. *Phys Rev D*, 2009,79: 124026.
- [12] Chen D Y, Yang H T, Zu X T. Area spectra of near extremal black holes. *Eur Phys J C*, 2010, 69: 289-292 .
- [13] Kwon Y, Nam S. Area spectra of the rotating BTZ black hole from quasinormal modes. *Class Quantum Grav*, 2010, 27: 125007;
Myung Y S. Area spectrum of slowly rotating black holes. arXiv:hep-th/1003.3519.
- [14] Lopez-Ortega A. Area spectrum of the d-dimensional Reissner-Nordström black hole in the small charge limit. *Class Quantum Grav*, 2011, 28:035009.
- [15] Daghighi R G, Green M D, Mulligan B W. Asymptotic spectrum of Kerr black holes in the small angular momentum limit. *Phys Rev D*, 2011, 83: 044001.
- [16] Majhi B R, Vagenas E C. Black hole spectroscopy via adiabatic invariance. *Phys Lett B*, 2011,701: 623-625.
- [17] Liu L, Pei S Y. Sommerfeld 's quantum condition of action and the spectra of quantum Schwarzschild black hole. *Chin Phys Lett*, 2004, 21: 1887-1889.
- [18] Banados M, Teitelboim C, Zanelli J. *Phys Rev Lett*, 1992, 69: 1849-1851.
- [19] Medved A J M. Radiation via tunnelling in the charged BTZ black hole. *Class Quant Grav*, 2002,19: 589-598;
Hortacsu M, Ozcelik H T, Yapiskan B, Properties of solutions in 2 + 1 dimensions. *Gen Rel Grav*, 2003, 35: 1209-1221.
- [20] Vagenas E C. Semiclassical corrections to the Bekenstein-Hawking entropy of the BTZ black hole via self-gravitation. *Phys Lett B*, 2002,533: 302-306;
Vagenas E C. Two-dimensional dilatonic black holes and Hawking radiation. *Mod Phys Lett A*, 2002, 17: 609-618.
- [21] Liu W B. New coordinates for BTZ black hole and Hawking radiation via tunnelling. *Phys Lett B*, 2006, 634: 541-544;
He X K, Liu W B. Modified Hawking radiation in a BTZ black hole using Damour-Ruffini method. *Phys Lett B*, 2007, 653: 330-334.
- [22] Wu S Q, Jiang Q Q. Remarks on Hawking radiation as tunneling from the BTZ black holes. *J High Energy Phys*, 2006, 03: 079.
- [23] Zhang J Y. Entropy correction of BTZ black holes in a tunneling frame work. *Sci China Ser G-Phys Mech Astron*, 2010,53(8):1427-1433.
- [24] Gibbons G W, Perry M J. Black holes and thermal Green functions. *Proc R Soc Lond*, 1978, A358: 467-494.
- [25] Landau L D, Lifshitz E M, *Mechanics*, 3rd ed. London: Pergamon Press, 1976.138-157.